1 How it all began with the complex numbers

Our goal in this course is to study the basics of *Complex Analysis*, which is, informally speaking, is a mathematical discipline that studies the properties of *analytic* (or *holomorphic*, do not worry, we'll get to a precise definition in a due time) functions of complex argument $z \in \mathbf{C}$, where, as usual, notation $z \in \mathbf{C}$ means that variable z belongs to the set of all complex numbers $\mathbf{C}$ (another standard notation for the same set is $\mathbb{C}$, but I will stick in these notes to the former one). But before embarking on our main goal, we need to carefully understand *what* a complex number is. Yes, of course, the usual answer of anyone who already saw complex numbers, is as follows: a complex number z is an expression of the form $a + \mathrm{i}b$, where $a, b \in \mathbf{R}$ (recall that $\mathbf{R}$ is the set of *real* numbers), and i is *the imaginary unit* that has the characteristic property $\mathrm{i}^2 = -1$. But then I will ask: Still, what is "i"? Can we have some other (physical, geometric) meaning of this creature other than this very strange, unmotivated, and even bizarre property of being negative one if squared? There is a very simple answer to this question, but before providing it, I will give a short account of why something like $\mathrm{i}^2 = -1$ turned out to be necessary.

1.1 A few words about a couple of Italian mathematicians from fifteenth and sixteenth centuries

Complex numbers how we know them were introduced to solve cubic equations of the form

$$x^3 + ax^2 + bx + c = 0,$$

where a, b, c are real numbers.

For thousands of years people knew how to solve a quadratic equation

$$x^2 + px + q = 0 \quad \Longrightarrow \quad x_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2},$$

which may have either two (real) roots, or one real root, or no real roots at all depending on the expression $p^2 - 4q$, which is called the *discriminant*. Contrary to what some textbooks say, there was no issue with these three distinct possibilities, and especially with the fact that sometimes quadratic equation has no roots at all, since there is a simple geometric interpretation for this algebraic problem. Namely, let me rewrite my quadratic equation as

$$x^2 = -px - q,$$

and hence the roots of my equation geometrically are intersections of the quadratic parabola $y = x^2$ and the straight line $y = -px - q$. Clearly, no matter how I pick p, q and plot these two curves, it is only possible to have either two points of intersection, or one point, or no points at all (see Fig. 1).

It took much longer to figure our how to attack a general cubic equation. It seems that the first person, who actually was able to solve the so-called *depressed* cubic equation

$$x^3 + px + q = 0$$

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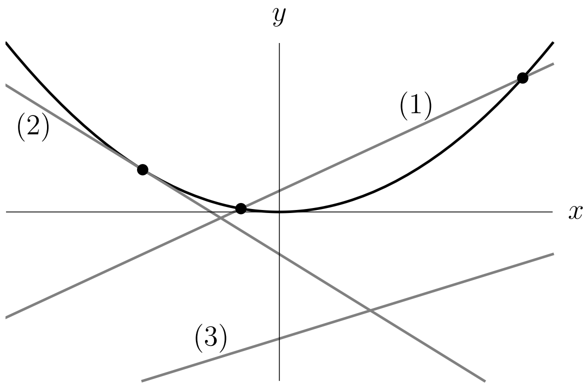

Figure 1: Geometric interpretation of three possible cases while solve a quadratic equation: (1) two roots, (2) one root, (3) no roots.

with only *positive* p and q was Italian mathematician Scipione del Ferro (1465–1526). He did not accept negative numbers and therefore did not think that he discovered something very general (e.g., he would consider the problem $x^3 = px + q$ to be a completely different one). He kept his discovery as a secret, since there was a tradition at the time to have mathematical competitions with monetary prices and hence this knowledge would allow him to make some cash. Only when he was dying he shared his secret with one of his students Antonio Maria Fior, who decided to capitalize on this knowledge. In particular, he found out about Niccolo Fontana (1500–1577), who actually was a very bright mathematician, and who claimed that he figured the way to solve the cubic equations of the form $x^2 + px^2 = q$. Fior belived that Fontana, who is nowadays more known under the nickname Tartaglia, was bluffing and challenged him. Tartaglia knew that Fior can solve depressed cubic equations and was able to discover the formula for the roots before the contest. So he was able to solve Fior's problems, whereas his own problems were too complicated to Fior, and hence won the contest.

Here in our story comes Girolamo Cardano (1501–1576), yet another Italian mathematician, who convinced Tartaglia to share his secret solution under the oath of not publishing it (that happened around 1539). Cardano kept his secret till 1545 when he found out that already del Ferro had the same method. He published these results about the cubic equation in his *Ars Magna* ("The Great Art") giving credit to both del Ferro and Tartaglia (who, by the way, was furious and challenged Cardano for contest, which Cardano refused). Ludovico Ferrary (1522–1565) was a student of Cardano, who accepted the challenge and won the contest (Tartaglia lost a lot of money and died in poverty; which is even more ironic, in the history of mathematics the formula that solves the cubic equation is nowadays known as *Cardano's formula* despite the fact that two people who must get the credit for it are del Ferro and Tartaglia). To jump ahead, in Cardano's formula under certain circumstances, one needs to use square roots of negative numbers. Cardano himself did not include any such explicit calculations in his book, and arguably the first person with an explicit example was Rafael Bombelli (1526–1572), who wrote a book on algebra that was published in 1572, and in which for the first time the rules how to work with complex numbers were formulated.

1.2 Cardano's formula

Let me start with a general cubic equation of the form

$$y^3 + ay^2 + by + c = 0,$$

where a, b, c are given (real) numbers. First, I make a substitution

$$y = x - \frac{a}{3},$$

which leads to

$$\begin{aligned} y^3 + ay^2 + by + c &= \\ \left(x - \frac{a}{3}\right)^3 + a\left(x - \frac{a}{3}\right)^2 + b\left(x - \frac{a}{3}\right) + c &= \\ x^3 + x\left(b - \frac{a^2}{3}\right) + \left(\frac{2a^3}{3^3} - \frac{ab}{3} + c\right) &= \\ x^3 - px - q = 0, \end{aligned}$$

where I introduced the new parameters

$$p = -\left(b - \frac{a^2}{3}\right), \quad q = -\left(\frac{2a^3}{3^3} - \frac{ab}{3} + c\right).$$

Therefore, my first conclusion is that *any* general cubic equation can be reduced to the depressed form

$$x^3 = px + q. \tag{1.1}$$

In order to solve (1.1), I put $x = s + t$, which leads to

$$\begin{aligned} (s+t)^3 = p(s+t) + q \quad &\Longrightarrow \\ s^3 + 3s^2t + 3st^2 + t^3 - ps - pt - q = 0, \quad &\Longrightarrow \\ s^3 + t^3 - q + (s+t)(3st - p) = 0. \end{aligned}$$

Now I make a bold move and reduce the above equation to two new ones:

$$s^3 + t^3 = q$$

and

$$3st = p \implies t = \frac{p}{3s}.$$

Using the previous equation I get

$$s^6 - qs^3 + \frac{p^3}{3^3} = 0,$$

which is a quadratic equation for the variable $h = s^3$:

$$h_{1,2} = \frac{q}{2} \pm \sqrt{\frac{q^2}{2^2} - \frac{p^3}{3^3}},$$

and since s and t are symmetric in my formulas and I must have $s^3 + t^3 = q$, I must choose for s, say, the sign plus and for t the sign minus, that is

$$x = s + t = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{2^2} - \frac{p^3}{3^3}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{2^2} - \frac{p^3}{3^3}}},$$

which is Cardano's formula (be careful, in the literature there are other, equivalent ways of writing it).

To get some practice with Cardano's formula, let's solve

$$x^3 = 6x + 6,$$

which leads to

$$x = \sqrt[3]{3 + \sqrt{\frac{6^2}{4^2} - \frac{6^3}{3^3}}} + \sqrt[3]{3 - \sqrt{\frac{6^2}{4^2} - \frac{6^3}{3^3}}} = \sqrt[3]{4} + \sqrt[3]{2},$$

which can be directly checked to be a legitimate root of our problem. Is it possible to have other roots? Yes, of course, but if we already know one root, we can always use the long division algorithm to reduce our problem to a quadratic equation, whose solution we know (make sure that you clearly understand the meaning of the previous phrase). So, the problem solved? Not so fast.

Look at

$$x^3 = 15x + 4,$$

which yields, using Cardano's formula,

$$x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}}.$$

I may say that since I need to extract square roots from negative numbers it indicates that my problem has no solution, as I explained when I presented the formula for the quadratic equation. But geometrically the situation now is very different! According to (1.1), my roots are the intersections of the cubic parabola $y = x^3$ and a straight line $y = px + q$. Clearly, no matter what p and q are, there must be at least one point of intersection (see Fig. 2). In other words: Any cubic equation has at least one (real) root.

For our specific example a few seconds of thought yields that the root we are looking for is $x = 4$. And it was actually Bombelli's ingenious idea, after looking at Cardano's formula, to assume that

$$4 = x = \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}} = 2 + n\sqrt{-1} + 2 - n\sqrt{-1} = 4,$$

where I need only to determine n.

For this idea to work I must have the rules of addition and multiplication of numbers that involve $\sqrt{-1}$. For addition it is quite natural to require

$$(a + b\sqrt{-1}) + (c + d\sqrt{-1}) = (a + c) + (b + d)\sqrt{-1}.$$

I need multiplication since we need to determine n using the property

$$(2 + n\sqrt{-1})^3 = 2 + 11\sqrt{-1}.$$

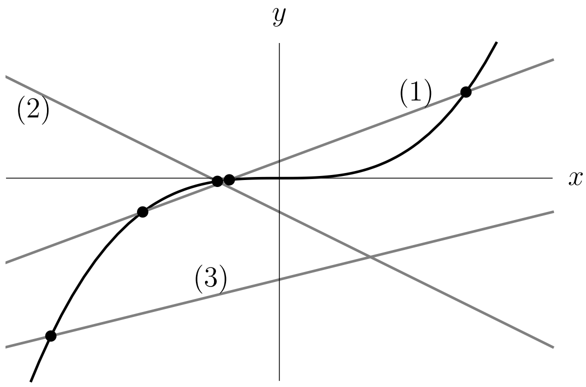

Figure 2: Geometrically solving cubic equation.

Again, using our usual rules,

$$(a+b\sqrt{-1})(c+d\sqrt{-1}) = (ac+bd\sqrt{-1}\sqrt{-1}) + (bc+ad)\sqrt{-1} = (ac-db) + (bc+ad)\sqrt{-1},$$

where I, following Bombelli, used a "natural" calculation $\sqrt{-1}\sqrt{-1} = \sqrt{(-1)^2} = -1$, which by no means obvious (as late as 1770 Leonard Euler argued that $\sqrt{-3}\sqrt{-2} = \sqrt{6}$, which is incorrect)! But if I stick to this rule, I can easily produce (check it) that

$$(2+\sqrt{-1})^3 = 2 + 11\sqrt{-1}, \quad (2-\sqrt{-1})^3 = 2 - 11\sqrt{-1},$$

and everything works out as required. Much later great Swiss mathematician Leonard Euler used the symbol i to denote the (positive) square root of -1, and therefore my arithmetic rules become

$$(a+\mathrm{i}b) + (c+\mathrm{i}d) = (a+c) + \mathrm{i}(b+d),$$
$$(a+\mathrm{i}b)(c+\mathrm{i}d) = (ac-bd) + \mathrm{i}(bc+ad),$$

which you all saw before.

After all these algebraic and historical details I can finally explain why the name "imaginary" was used. The main point here is that these numbers, which did not seem to have a simple physical meaning, allow us to find perfectly fine *real* numbers (look again at the figure with the cubic parabola), i.e., they are used as a not very well understood, but necessary intermediate step in calculations that lead to perfectly meaningful answers. For a very long time these numbers were perceived as such (e.g., in 1702 Leibnitz, one of the fathers of Calculus, described i as "that amphibian between existence and nonexistence.") Only by the end of eighteenth century it was realized that there was a perfectly fine concrete interpretation of complex numbers: viz., they are exactly points (or vectors) on the plane that can be added and multiplied following some certain natural rules. I will expand on this much more in the next lecture.

1.3 A digression: How to find the roots of polynomials

Any (well, almost) student knows how to solve a quadratic equation

$$ax^2 + bx + c = 0,$$

and almost no one remembers Cardano's formula for determining the roots of a cubic polynomial. To complicate the matter further, I would like to state that although for polynomials of degree 4 it is possible to find a formula similar to Cardano's one, for a general polynomial of degree five it is *impossible* to find such a formula, which is a very deep result (google Abel–Ruffini theorem). So what if one needs to determine roots of a polynomial of degree bigger then two? In general this must be done numerically, and there exist very efficient algorithms of doing this. In class, however, a student is often given a polynomial with *integer* coefficients:

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0,$$

where all a_j are integers. For such equations the following theorem holds:

Theorem 1.1. *Assume that $a_0, a_n \neq 0$ and all other coefficients are integers. Then any rational solution of the above equation, which can be written as $x = p/q$, p and q are coprime, must satisfy:*

1. *p divides a_0;*
2. *q divides a_n.*

Problem 1.1. Show that $2x^3 + x - 1$ has no rational roots.

Problem 1.2. Assume that $a_n = 1$ and formulate a useful corollary of the theorem.

Problem 1.3. Prove this theorem (it requires some basic knowledge of Number theory).